Surface Reconstruction via Smoothest Restricted Range Approximation

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Abstract. In this paper we describe the application of smoothest restricted range approximation to surface reconstruction. Smoothest restricted range approximation is the minimization of a quadratic functional measuring smoothness over all functions in a suitable Hilbert space satisfying a given finite set of constraints. This setting includes both specified smoothness penalties and compactly supported radial basis functions. Compared to competing greedy fitting techniques smoothest restricted range approximation offers better denoising and data compression.

§1. Introduction

In this paper we describe the application of smoothest restricted range approximation to surface reconstruction. Smoothest restricted range approximation is the minimization of a quadratic functional measuring smoothness over all functions in a suitable Hilbert space satisfying a given set of constraints. In a natural setting all solutions to the problem turn out to be radial basis functions (RBFs) of the form

$$s = q_{k-1} + \sum_{j=1}^{N} \lambda_j \Phi(\bullet - \mathbf{x}_j),$$

where q_{k-1} is a polynomial of low degree, $k-1$, and Φ is a fixed function determined by the measure of smoothness. The basic function Φ may be either globally or compactly supported. Compared to competing greedy fitting techniques, smoothest restricted range fitting offers better denoising and data compression.

The layout of the paper is as follows. In Section 2 we give a self contained description of the reproducing kernel Hilbert space setting for minimal energy interpolation and smoothest restricted range approximation

Fig. 1. LIDAR scans and smoothest restricted range reconstruction of the Statue of Liberty and a Buddha statue.

problems. In Section 3 we discuss the application of smoothest restricted range fitting to surface reconstruction. We present numerical results in the form of pictures and tables showing the advantages of the method for this application.

§2. A Hilbert Space Setting for Minimum Energy Interpolation and Smoothest Restricted Range Approximation

In this section we consider a Hilbert space setting for minimal energy interpolation and smoothest restricted range interpolation. Minimal energy interpolation has previously been been considered in many papers for example Duchon [6], Schaback [12, 13] and Light and Wayne [10]. The smoothest restricted range interpolation problem was first considered by Ritter [11] and Kimeldorf and Wahba [8] for splines in $\mathcal{R}^1$. Dubrule and Kostov [5], and Villalobos and Wahba [14] give treatments for functions in $\mathcal{R}^d$. This problem concerns finding a function s which minimizes a semi-inner product measuring energy subject to satisfying a finite number of inequality contraints. For example the constraints on the fitted function s may be that $\ell_i \leq s(\mathbf{x}_i) \leq u_i$, for $1 \leq i \leq N$. The smoothest restricted range approximation problem is closely connected to the theory of support vector machines.

We give a concise self contained treatment of the theory of smoothest restricted range approximations. Our treatment is deliberately elementary and avoids the use of the Karush-Kuhn-Tucker conditions. The main result is that the smoothest restricted range approximation problem for both a quadratic penalty and for compactly supported RBFs, leads to a quadratic programming problem (QPP), see equation (12). The solution is an RBF with centres only where the constraints are active (see Theorem 2). As a consequence, smoothest restricted range fits often involve orders of magnitude less RBF centres than there are data points.

The standard setting for smoothest interpolation problems is a real Hilbert space $\mathcal{H}$ with a semi-inner product $\langle f, g\rangle$ viewed as quadratic measure of smoothness. This semi-inner product is assumed to have polynomial kernel π^d_{k-1}. Then given a set of points $\{x_1, x_2, \ldots, x_\ell\}$ such that the Lagrange polynomials for this set $\{p_1, p_2, \ldots, p_\ell\}$ form a basis for π^d_{k-1}, the polynomials of degree not exceeding $k-1$ on $\mathcal{R}^d$, an inner product for $\mathcal{H}$ is defined by

$$(f, g) = \sum_{i=1}^{\ell} f(x_i)g(x_i) + \langle f, g\rangle. \tag{1}$$

The typical example of this setup is the Beppo-Levi space associated with the k-harmonic smoothness penalty in $\mathcal{R}^d$. Here the quadratic penalty is

$$\langle f, g\rangle = \sum_{|\alpha|=k} \frac{k!}{\alpha_1!\alpha_2!\alpha_3!\ldots\alpha_d!} \int_{\mathcal{R}^d} D^\alpha f(\mathbf{x}) D^\alpha g(\mathbf{x}) d\mathbf{x}. \tag{2}$$

The associated semi-norm $|f| = \sqrt{\langle f, f\rangle}$ clearly measures smoothness and has kernel π^d_{k-1}. Some further examples are given below.

Further, it is usually assumed that point evaluations are continuous functionals on the Hilbert space. This assumption implies that the Hilbert space is a reproducing kernel Hilbert space (see for example Davis [4]). That is, there exists a unique kernel $K : \mathcal{R}^d \times \mathcal{R}^d \to \mathcal{R}$, with the following properties

(i) $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{R}^d$.

(ii) $K(\bullet, \mathbf{x}) \in \mathcal{H}$ for all $\mathbf{x} \in \mathcal{R}^d$.

(iii) K is strictly positive definite. That is, for any choice of distinct points $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathcal{R}^d$ and numbers $a_1, \ldots, a_m \in \mathcal{R}$, $\sum_{i,j=1}^m a_i a_j K(\mathbf{x}_i, \mathbf{x}_j) \geq 0$, with equality only when $\mathbf{a} = \mathbf{0}$.

(iv) $f(\mathbf{y}) = (f, K(\bullet, \mathbf{y}))$ for all $f \in \mathcal{H}$ and $\mathbf{y} \in \mathcal{R}^d$.

Further assumptions on the form of the smoothness penalty $\langle f, g\rangle$ (see Schaback [12, 13], Light and Wayne [10], and Levesley and Light [9]) imply that associated with the Hilbert space is a radial strictly conditionally positive definite function of order k, Φ such that the reproducing kernel can be written as

$$K(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \sum_{j=1}^{\ell} p_j(\mathbf{y})\Phi(\mathbf{x}-\mathbf{x}_j) - \sum_{i=1}^{\ell} p_i(\mathbf{x})\Phi(\mathbf{x}_i-\mathbf{y}) \qquad (3)$$
$$+ \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} p_i(\mathbf{x})p_j(\mathbf{y})\Phi(\mathbf{x}_i-\mathbf{x}_j) + \sum_{i=1}^{\ell} p_i(\mathbf{x})p_i(\mathbf{y}), \quad \mathbf{x},\mathbf{y}\in\mathcal{R}^n.$$

Let us mention a few examples of this structure. The most prominent example is the polyharmonic spline, where the quadratic penalty is given by equation (2) and, for $k > d/2$, the corresponding function Φ is the fundamental solution of the equation $\triangle^k\Phi = \delta$. Another example is the Sobolev penalty

$$(f,g) = \langle f,g\rangle = \int_{\mathcal{R}^d} \mathcal{F}f(\boldsymbol{\omega})\overline{\mathcal{F}g}(\boldsymbol{\omega})\left(1+|\boldsymbol{\omega}|^2\right)^j d\boldsymbol{\omega},$$

where $\mathcal{F}$ is the Fourier transform operator. Schaback [12, 13] shows that for this penalty Φ is a multiple of the Bessel kernel

$$G_{d,\alpha}(\mathbf{x}) = \frac{1}{\pi^{d/2}2^{(d+\alpha-2)/2}\Gamma(\alpha/2)} K_{(d-\alpha)/2}(|\mathbf{x}|)|x|^{(\alpha-d)/2}.$$

where $\alpha = 2j > d$. Another example is the penalty

$$(f,g) = \langle f,g\rangle = \int_{\mathcal{R}^d} \exp(|\boldsymbol{\omega}|^2)\mathcal{F}f(\boldsymbol{\omega})\overline{\mathcal{F}g}(\boldsymbol{\omega})d\boldsymbol{\omega}$$

discussed by Levesley and Light [9] for which $\Phi(\mathbf{x}) = c\exp(-|\mathbf{x}|^2/4)$ and $k = 0$. Finally, consider the case of compactly supported radial basis functions based upon a strictly positive definite radially symmetric basic function Φ. Then for $s = \sum_{i=1}^N \lambda_i\Phi(\bullet - \mathbf{x}_i)$ the energy functional is $(s,s) = \langle s,s\rangle = \sum_{i,j=1}^N \lambda_i\lambda_j\Phi(\mathbf{x}_i-\mathbf{x}_j)$.

The reader should note that in solving the RBF interpolation equations one can just as easily use a multiple of Φ as the precise function Φ associated with the smoothness penalty. For example in forming 3D biharmonic spline interpolants we can use $\Phi = |\bullet|$ rather than $\Phi = -|\bullet|/(8\pi)$. However, the same is not true for many of the equations in the current paper. In particular equations (3), (4), (5) and (12) only hold for correctly scaled and signed Φ's (otherwise these equations would require additional multiplicative constants).

The reproducing kernel property implies that for functions s of the special form $s(\bullet) = \sum_{i=1}^N a_iK(\bullet,\mathbf{x}_i)$

$$(s,s) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_ia_jK(\mathbf{x}_i,\mathbf{x}_j) = \mathbf{a}^T\mathsf{K}\mathbf{a}, \qquad (4)$$

where K is the $N \times N$ matrix with ij entry $K(\mathbf{x}_i, \mathbf{x}_j)$. A somewhat involved calculation then shows that for

$$s(\bullet) = \sum_{j=1}^{N} a_j K(\bullet, \mathbf{x}_j) = q_{k-1}(\bullet) + \sum \lambda_j \Phi(\bullet - \mathbf{x}_j),$$

the inner product,

$$(s, s) = \mathbf{a}^T \mathsf{K} \mathbf{a} = \boldsymbol{\lambda}^T \mathsf{A} \boldsymbol{\lambda} + \sum_{j=1}^{\ell} s(\mathbf{x}_j)^2,$$

where $\mathsf{A}_{ij} = \Phi(\mathbf{x}_i - \mathbf{x}_j)$. Therefore using equation (1)

$$\langle s, s \rangle = \boldsymbol{\lambda}^T \mathsf{A} \boldsymbol{\lambda}. \tag{5}$$

Thus the continuous quadratic measure of energy $\langle s, s \rangle$ reduces to the discrete form $\boldsymbol{\lambda}^T \mathsf{A} \boldsymbol{\lambda}$ in the case of an RBF of form

$$s(\bullet) = q_{k-1}(\bullet) + \sum \lambda_j \Phi(\bullet - \mathbf{x}_j), \tag{6}$$

where

$$q_{k-1} = c_1 p_1 + c_2 p_2 + \cdots + c_\ell p_\ell \in \pi_{k-1}^d, \tag{7}$$

and the coefficients $\boldsymbol{\lambda}$ are "orthogonal to π_{k-1}^d" in the sense that

$$\sum_{j=1}^{N} \lambda_j q(\mathbf{x}_j) = 0, \qquad \text{for all} \quad q \in \pi_{k-1}^d. \tag{8}$$

We have now assembled enough machinery to produce several known results about interpolation and smoothest restricted range approximation. The results about smoothest restricted range approximation would normally be deduced from the Karush Kuhn Tucker optimality conditions but we give an alternative elementary approximation theoretic development.

Theorem 1. (Minimum energy interpolation.) *Assume that $\mathcal{H}$ is a Hilbert space with energy semi-inner product $\langle \bullet, \bullet \rangle$, strictly conditionally positive definite function Φ of order k, and reproducing kernel K as above. Given N points $\mathcal{X} = \{\mathbf{x}_i,\ 1 \le i \le N\}$ of $\mathcal{R}^d$, unisolvent for π_{k-1}^d, and N corresponding real values f_i there is a unique function s in $\mathcal{H}$ which minimizes the energy semi-inner product $|g|^2 = \langle g, g \rangle$ over all functions $g \in \mathcal{H}$ satisfying the interpolation conditions*

$$g(\mathbf{x}_i) = f_i, \quad 1 \le i \le N.$$

This function is an RBF of the form

$$s = \sum_{j=1}^{N} a_j K(\bullet, \mathbf{x}_j) = q_{k-1} + \sum_{j=1}^{N} \lambda_j \Phi(\bullet - \mathbf{x}_j). \tag{9}$$

where $q_{k-1} \in \pi_{k-1}^d$ *and the weights* λ_j *satisfy (8).*

Proof: The positive definiteness of the matrix K defined in (4) implies that we can find an interpolant s of the form of the Theorem by solving the system $\mathsf{K}\mathbf{a} = \mathbf{f}$. Let g be any other interpolant from $\mathcal{H}$ and write it in the form $g = s + r$. The reproducing kernel property implies $(r, s) = \sum_{j=1}^{N} a_j r(x_j)$. However, in order that the interpolation conditions are satisfied by g, r must be zero on $\mathcal{X}$. Hence $(r, s) = 0$ implying $(g, g) = (s, s) + (r, r)$. Therefore there is a unique interpolant $g \in \mathcal{H}$ minimizing (g, g) and it is s. Since all interpolants have the same value of $\sum_{i=1}^{\ell} (g(\mathbf{x}_i))^2$ it follows that s minimizes the energy semi-inner product $|g|^2 = \langle g, g\rangle$ as well. □

As a consequence of Theorem 1 finding the minimum energy interpolant in the Hilbert space reduces to finding the coefficients of an RBF interpolant. That is it reduces to solving the system

$$\begin{pmatrix} \mathsf{A} & \mathsf{P} \\ \mathsf{P}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}, \tag{10}$$

for the coefficients $\boldsymbol{\lambda}$ and $\mathbf{c}$ where

$$\begin{aligned} A_{i,j} &= \Phi(\mathbf{x}_i - \mathbf{x}_j), & i, j &= 1, \dots, N, \\ P_{i,j} &= p_j(\mathbf{x}_i), & i &= 1, \dots, N, \quad j = 1, \dots, \ell. \end{aligned}$$

Theorem 2. (Smoothest restricted range approximation.) *Assume that* $\mathcal{H}$ *is a Hilbert space with energy semi-inner product* $\langle \bullet, \bullet \rangle$*, strictly conditionally positive definite function* Φ *of order* k*, and reproducing kernel* K *as above. Given a set of nodes* $\mathcal{X} = \{\mathbf{x}_i\}_{i=1}^N \subset \mathcal{R}^d$*, unisolvent for* π_{k-1}^d*, and a set of function bounds* $\{(\ell_i, u_i) : \ell_i \le u_i\}_{i=1}^N$ *where* ℓ_i *may be* $-\infty$ *and* u_i *may be* $+\infty$*:*

(i) There is a function $g \in \mathcal{H}$ *which minimizes the energy semi-inner product* $|g|^2 = \langle g, g \rangle$ *over all elements of* $\mathcal{H}$ *satisfying the restricted range constraints*

$$\ell_i \le g(\mathbf{x}_i) \le u_i, \qquad 1 \le i \le N.$$

Furthermore, any solution g *is an RBF with centres* $\mathcal{X}$*. That is, it has the form*

$$s = \sum_{j=1}^{N} a_j K(\bullet, \mathbf{x}_j) = q_{k-1} + \sum_{j=1}^{N} \lambda_j \Phi(\bullet - \mathbf{x}_j). \tag{11}$$

where $q_{k-1} \in \pi^d_{k-1}$ and the weights λ_j satisfy (8).

(ii) A solution can be found by solving the QPP (12).

(iii) Any two solutions g differ at most by a polynomial of degree $k-1$.

(iv) If $\mathcal{X}$ remains unisolvent for π^d_{k-1} when any one node $\mathbf{x}_j$ is removed then the RBF weights corresponding to non binding constraints are zero.

Proof: *Proof of (i) and (ii)* Firstly observe that the problem is feasible, as for each index i there is a real number f_i with $\ell_i \leq f_i \leq u_i$ and one can then interpolate the data $\{(\mathbf{x}_i, f_i)\}$, for example with an RBF. Given any feasible function $g \in \mathcal{H}$ for the optimisation problem the minimum energy interpolation theorem, Theorem 1, implies that either g is an RBF with centres $\mathcal{X}$ or else the RBF which interpolates to data $\{(\mathbf{x}_i, g(\mathbf{x}_i))\}$ has lower energy. Thus, the problem of finding a smoothest restricted range interpolant can be reduced to that of minimizing the quadratic energy over all RBFs s satisfying the constraints. Further, the linear constraints on the values of the RBF can be expressed as linear inequalities on the coefficients. Thus, our problem becomes

$$\text{minimize}_{[\boldsymbol{\lambda}^T \mathbf{c}^T] \in \mathcal{R}^{N+\ell}} \quad \boldsymbol{\lambda}^T \mathsf{A} \boldsymbol{\lambda} \tag{12a}$$

$$\text{subject to} \quad \mathsf{A}\boldsymbol{\lambda} + \mathsf{P}\mathbf{c} \leq \mathbf{u} \tag{12b}$$

$$\text{and} \quad -(\mathsf{A}\boldsymbol{\lambda} + \mathsf{P}\mathbf{c}) \leq -\boldsymbol{\ell} \tag{12c}$$

$$\text{and} \quad \mathsf{P}^T \boldsymbol{\lambda} = \mathbf{0}. \tag{12d}$$

This is an inequality constrained QPP. It is well known that a convex quadratic program which is bounded below necessarily has a solution. That is, there is a finite feasible point where the objective function achieves its infimum. See for example Wolfe [15]. Thus there is at least one solution to our original smoothest restricted range interpolation problem and it is an RBF.

Proof of (iii) We consider the Hilbert space of equivalence classes $\mathcal{H}/\pi^d_{k-1}$ with inner product $\langle \bullet, \bullet \rangle$ applied to any two representatives. Let s_1 and s_2 be any two solutions of the smoothest restricted range approximation problem with energy E. Then considering them as representatives of equivalence classes in $\mathcal{H}/\pi^d_{k-1}$ we have $\langle (s_1 + s_2)/2, (s_1 + s_2)/2 \rangle = (1/4)(\langle s_1, s_1 \rangle + \langle s_2, s_2 \rangle + \langle s_1, s_2 \rangle + \langle s_2, s_1 \rangle)$. Applying Cauchy-Schwarz this is less than E unless $\langle s_1, s_2 \rangle = E$. Hence, since equality holds in the Cauchy-Schwarz inequality only for parallel vectors/functions, it follows that the two equivalence classes are equal. That is in the space $\mathcal{H}$, $s_1 = s_2$ modulo an element of π^d_{k-1}.

Proof of (iv) Suppose s is a solution and for index j, $\ell_j < s(\mathbf{x}_j) < u_j$. Further, suppose that the coefficient λ_j in the expansion (11) of s is non-zero. Then let s_1 be the RBF interpolant to s at all points of $\mathcal{X}$ except

$\mathbf{x}_j$. The minimum energy interpolation Theorem, Theorem 1, implies s_1 has less energy than s. Therefore for some sufficiently small positive θ the RBF $(1-\theta)s + \theta s_1$ satisfies the constraints and has less energy than s. Therefore s is not a solution. Contradiction. $\square$

One standard approach (see [7]) to solving such inequality constrained QPPs is an active set strategy where one moves through a sequence of equality constrained QPPs, trying to determine which inequality constraints are active at a global minimum point. In our case the equality constrained sub-problems are of the form: *Find the minimum energy interpolant taking certain values at a subset of the nodes* $\{\mathbf{x}_i\}$. Either from the QPP itself, or Theorem 1, these minimum energy interpolation subproblems are RBF interpolation problems, which can be solved efficiently by methods such as those described in [1] and [2]. At the time of writing we recommend solving such equality constrained subproblems with an iterative method such as domain decomposition preconditioned FGMRES.

§3. Application to Surface Reconstruction

In this section we consider the task of fitting a surface to a LIDAR or laser scan. Typically this task is organised into three stages

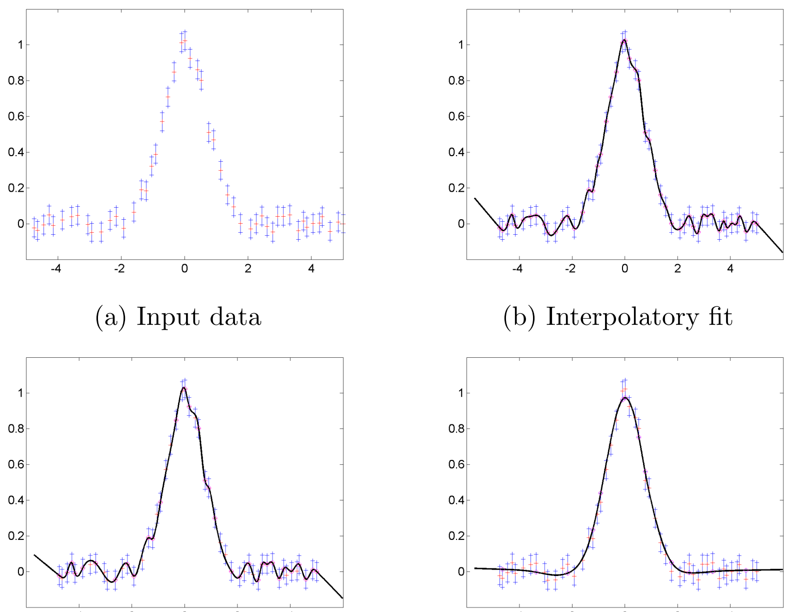
(i) *Producing signed distance data.* The original data is a collection of points $\{\mathbf{x}_i\}$ approximately on the surface. Produce from it a larger set of data $\{(\mathbf{x}_i, f_i)\}$ where f_i is zero for points $\mathbf{x}_i$ "on the surface", and an approximate signed distance to the surface for other points.

(ii) *Approximation of signed distance data.* Interpolate, or approximate, the set $\{(\mathbf{x}_i, f_i)\}$ with a function $s : \mathcal{R}^3 \to \mathcal{R}$ bearing in mind that the data is usually noisy, and there is usually a high degree of oversampling.

(iii) *Isosurface extraction.* Given the approximation s construct an approximate isosurface corresponding to the set of points where $s(\mathbf{x}) = 0$. The approximate isosurface will be constructed by some procedure such as marching cubes or marching tetrahedra.

[2, 3] describe various aspects of this procedure. Once the polygonal isosurface is formed it can be used for viewing the object or to drive a 3D printer.

In the current paper we concentrate on the middle of the three steps above. In particular, this section considers the use of smoothest restricted range approximation (discussed in Section 2) to form the approximation s. We use biharmonic splines associated with the quadratic penalty, or



(a) Input data

(b) Interpolatory fit

(c) Greedy fit

(e) Smoothest restricted range

Fig. 2. Smoothest restricted range approximation and other strategies compared in the case of cubic spline fitting.

energy semi-inner product,

$$< s, s >= |s|^2 = \int_{\mathcal{R}^3} \left(\frac{\partial^2 s(\mathbf{x})}{\partial x^2} \right)^2 + \left(\frac{\partial^2 s(\mathbf{x})}{\partial y^2} \right)^2 + \left(\frac{\partial^2 s(\mathbf{x})}{\partial z^2} \right)^2 + 2 \left(\frac{\partial^2 s(\mathbf{x})}{\partial x \partial y} \right)^2 + 2 \left(\frac{\partial^2 s(\mathbf{x})}{\partial x \partial z} \right)^2 + 2 \left(\frac{\partial^2 s(\mathbf{x})}{\partial y \partial z} \right)^2 d\mathbf{x}, \quad (13)$$

because of their attractive hole filling properties, [2, 3]. However, as discussed in Section 2, many other choices of quadratic penalty are possible.

In step (ii) above, and in many other practical problems, exact interpolation is not required. Typically we are given function values $\{f_i\}_{i=1}^N \subset \mathcal{R}$ and a bound ϵ and it is desired to find a function s with low energy such that $|s(\mathbf{x}_i) - f_i| \leq \epsilon$, for $1 \leq i \leq N$. Often this problem is solved by a greedy algorithm in which one interates: interpolating on a subset, checking the residual on the whole set, and then updating the subset with points on which the error is large. The process stops when the maximum residual does not exceed ϵ. Unfortunately, a greedy algorithm fit with

LIDAR scan	Number of interpolation nodes	Object size	Tolerance (ϵ)	Number of RBF centers	Isosurface resolution
Liberty	3,360,300	76m	0.01m	402,118	0.05m
Buddha	725,250	11m	0.005m	96,259	0.01m
St Marys	598,758	39m	0.005m	281,943	0.05m

Tab. 1. Results for smoothest restricted range fits to LIDAR scans.

nonzero tolerance has no energy minimisation characterisation. Such a greedy algorithm fit will usually pick up the overall tendencies first, and the high frequency noise last. However, this "smoothness" is only a fortunate side effect rather than something forced. Close examination of greedy fits often reveals unnecessary high frequency "noise". Hence we are led to force optimal smoothness by seeking instead a smoothest restricted range approximation as discussed in Section 2.

Consider a simple test case fitting a function g, by minimizing $\int_{-\infty}^{\infty}(g''(x))^2dx$ over all functions satisfying the constraints. Then $\Phi(x)$ is a multiple of $|\bullet|^3$ and $k = 2$. Figure 2 compares a minimum energy interpolant, a greedy algorithm fit, and a smoothest restricted range approximation for some artificial data. In this example, and typically, the greedy algorithm fit is visibly smoother than the exact interpolant and the smoothest restricted range fit is smoother still.

In the Figures 1 and 3 we give results from applying this method with the biharmonic penalty (13) to Cyra LIDAR scans of very large objects. Table 1 gives the details of these fits. The fitting tolerances chosen in these figures were based on the quoted accuracy of the scanner, 6mm, and ignored other potential noise sources such as inter-scan registration errors. Often the level of detail required for such large objects would allow the fitting tolerance to be relaxed. Although for these examples we have specified a global tolerance ϵ, we could just as easily have specified a different tolerance ϵ_i at each point $\mathbf{x}_i$.

The first two subfigures of Figure 1 show a cloud of points corresponding to the Statue of Liberty and the resulting isosurface. This raw point cloud has $1,930,206$ points. The addition of off surface points results in an approximation problem with $3,360,300$ data points. Fortunately the restricted range strategy, even with a very tight error tolerance, reduces the number of centers in the fitted RBF to $402,118$.

As well as the large LIDAR scans discussed above the smoothest restricted range strategy has been successfully applied to many hundreds of laser scans arising from medical applications, see *www.aranz.com/#hanger*. On the basis of this experience we feel that smoothest restricted range approximation with the biharmonic penalty is an excellent approximation strategy for the surface reconstruction application.

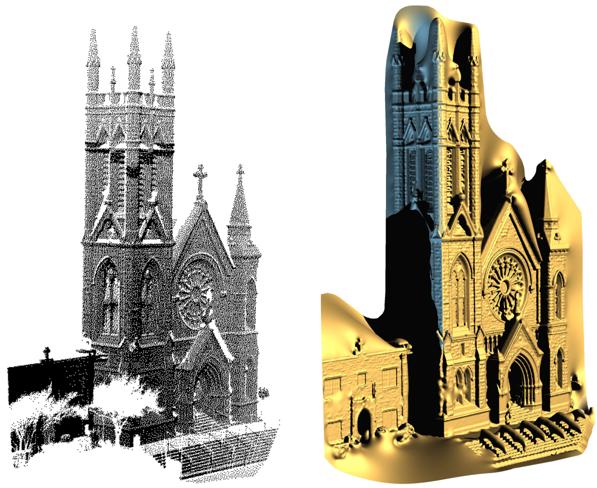


Fig. 3. LIDAR scan and smoothest restricted range reconstruction of St Marys church, LA. Foreground trees create occlusions in this scan.

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