Supplementary material: Proof of formula (39).

In this section we prove a lemma that implies formula (39) of the paper, namely

$$C_\nu(n) \;=\; \sum_{k=0}^{\nu} |\alpha_{\nu,k}(n)| \;=\; \frac{(2\nu-1)!!\,2^\nu}{(2n-1)(2n-5)\cdots(2n-4\nu+3)}, \qquad n \geq 2\nu-1, \tag{39}$$

The result (39) follows from the lemma below because, if we pick $m=\nu$ and $x=n$, then equations (26) and (73) with $n \geq 2\nu-1$ imply $|\alpha_{\nu,k}| = (2\nu-1)!!\,\beta_{m,k}(x)$, $k=0,1,\ldots,m$.

Lemma 8.1. *Let m be a positive integer and x be any real number such that $2x-4j+3 \neq 0$, for all integers j with $1 \leq j \leq m$. Let x be any real number such that $2x-2\ell+1 \neq 0$, for all integers ℓ in the interval $[1, 2m-1]$. Let $\beta_{m,k}(x)$ be the product*

$$\beta_{m,k}(x) \;=\; \binom{m}{k} \prod_{j=0,\, j\neq k}^{m} \frac{1}{2x-2j-2k+1}, \qquad k=0,1,\ldots,m. \tag{73}$$

Then

$$\sum_{k=0}^{m} \beta_{m,k}(x) \;=\; \frac{2^m}{\prod_{j=1}^{m}(2x-4j+3)}. \tag{74}$$

Proof. Our proof is by induction on m, beginning with $m=1$. In this case, the definition (73) provides the identity

$$\beta_{1,0}(x)+\beta_{1,1}(x) \;=\; \frac{1}{2x-1}+\frac{1}{2x-1} \;=\; \frac{2}{2x-1}, \tag{75}$$

so formula (74) holds. For the general step of the induction, we assume that formula (74) is correct when m is reduced by one, and we prove that it holds for the current m, where m is any integer that satisfies $m \geq 2$. The inductive hypothesis yields the identity

$$\frac{2^m}{\prod_{j=1}^{m}(2x-4j+3)} \;=\; \frac{2}{2x-4m+3} \sum_{k=0}^{m-1} \beta_{m-1,k}(x). \tag{76}$$

Therefore, letting $\beta_{m-1,m}(x)$ be zero, it is sufficient to establish the equation

$$\sum_{k=0}^{m} \left(\beta_{m,k}(x) - \frac{2}{2x-4m+3}\,\beta_{m-1,k}(x) \right) \;=\; 0. \tag{77}$$

We deduce from the definition (73) that the term inside the large brackets of equation (77) has the value

$$\begin{aligned}
&\frac{(m-1)!}{k!\,(m-k)!}\;\frac{m\,(2x-4m+3)-2\,(m-k)\,(2x-2k-2m+1)}{(2x-4m+3)\prod_{j=0,\,j\neq k}^{m}(2x-2j-2k+1)} \\
&\quad=\; \frac{(m-1)!}{k!\,(m-k)!}\;\frac{(4k-2m)\,x-4k^2+2k+m}{(2x-4m+3)\prod_{j=0,\,j\neq k}^{m}(2x-2j-2k+1)} \\
&\quad=\; (2x-4m+3)^{-1}\,\gamma_{m,k}(x),
\end{aligned} \tag{78}$$

which defines $\gamma_{m,k}(x)$ for $k=0,1,\ldots,m$, the choice $\beta_{m-1,m}(x)=0$ being admissible, because the contribution to $\gamma_{m,k}(x)$ from $\beta_{m-1\,k}(x)$ includes the factor $(m-k)$. We see that each function $\gamma_{m,k}(x)$, $x\in\mathcal{R}$, is a rational polynomial with a linear numerator, and that we have to establish $\sum_{k=0}^{m}\gamma_{m,k}(x)=0$.

For $\ell=0,1,\ldots,m-1$, we are going to calculate $\sum_{k=0}^{\ell+1}\gamma_{m,k}(x)$ by adding $\gamma_{m,\ell+1}(x)$ to $\sum_{k=0}^{\ell}\gamma_{m,k}(x)$. If equation (77) were true, then the rational functions $\sum_{k=0}^{\ell}\gamma_{m,k}(x)$ and $\sum_{k=\ell+1}^{m}\gamma_{m,k}(x)$, $x\in\mathcal{R}$, would have the same singularities. Hence $\sum_{k=0}^{\ell}\gamma_{m,k}(x)$, $x\in\mathcal{R}$, would be a polynomial in x divided by the product $\prod_{j=1}^{m}(2x-2j-2\ell+1)$. Thus the calculated partial sums take convenient forms, beginning with the expression

$$\sum_{k=0}^{0}\gamma_{m,k}(x) \;=\; \gamma_{m,0}(x) \;=\; \frac{1-2x}{\prod_{j=1}^{m}(2x-2j+1)}. \tag{79}$$

Our work will provide the values

$$\sum_{k=0}^{\ell}\gamma_{m,k}(x) \;=\; \frac{(m-1)!}{\ell!\,(m-\ell)!}\,\frac{(m-\ell)\,(4\ell+1-2x)}{\prod_{j=1}^{m}(2x-2j-2\ell+1)}, \qquad \ell=0,1,\ldots,m. \tag{80}$$

Then the factor $(m-\ell)$ in the numerator gives $\sum_{k=0}^{m}\gamma_{m,k}(x)=0$ as required.

It remains to establish formula (80) by induction. Equation (79) shows that it is true in the case $\ell=0$. Therefore we assume that equation (80) holds for any integer ℓ from $[0,m-1]$, and we add $\gamma_{m,\ell+1}(x)$ to expression (80), which gives the sum

$$\sum_{k=0}^{\ell+1}\gamma_{m,k}(x) \;=\; \frac{(m-1)!}{(\ell+1)!\,(m-\ell-1)!}\,\frac{\eta}{\prod_{j=0}^{m}(2x-2j-2\ell-1)}, \tag{81}$$

where η is the expression

$$\begin{aligned}\eta \;&=\; (\ell+1)\,(4\ell+1-2x)\,(2x-2\ell-2m-1)\\ &\qquad\qquad + \{(4\ell+4-2m)\,x - 4\ell^2 - 6\ell - 2 + m\}\,(2x-4\ell-3)\\ &=\; (m-\ell-1)\,(4\ell+5-2x)\,(2x-2\ell-1).\end{aligned} \tag{82}$$

The first two factors in the last line are needed by the numerator of expression (80) when ℓ is replaced by $\ell+1$, while the $(2x-2\ell-1)$ factor of η removes the leading singularity of the rational function (81). Thus formula (80) remains true when ℓ is increased by one. It follows from remarks made earlier that the analysis is complete. □

Supplementary material: Tables of associated Legendre functions, inner functions, and outer functions.

Tables of associated Legendre functions, inner functions and outer functions as normalised in the paper *Fast evaluation of polyharmonic splines in 3-dimensions* follow.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$m=4$					$105\sin^4\theta$
$m=3$				$-15\sin^3\theta$	$-105\sin^3\theta\cos\theta$
$m=2$			$3\sin^2\theta$	$15\sin^2\theta\cos\theta$	$\{\frac{105}{2}\cos^2\theta-\frac{15}{2}\}\sin^2\theta$
$m=1$		$-\sin\theta$	$-3\sin\theta\cos\theta$	$-\frac{3}{2}\sin\theta\{5\cos^2\theta-1\}$	$-\frac{5}{2}\sin\theta\cos\theta\{7\cos^2\theta-3\}$
$m=0$	$P_0=1$	$P_1(\cos\theta)=\cos\theta$	$P_2(\cos\theta)=\frac{3}{2}\cos^2\theta-\frac{1}{2}$	$P_3(\cos\theta)=\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta$	$P_4(\cos\theta)=\frac{35}{8}\cos^4\theta-\frac{15}{4}\cos^2\theta+\frac{3}{8}$
$m=-1$		$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta\cos\theta$	$\frac{1}{8}\sin\theta\{5\cos^2\theta-1\}$	$\frac{7}{8}\cos^3\theta\sin\theta-\frac{3}{8}\sin\theta\cos\theta$
$m=-2$			$\frac{1}{8}\sin^2\theta$	$\frac{1}{8}\sin^2\theta\cos\theta$	$\{\frac{7}{48}\cos^2\theta-\frac{1}{48}\}\sin^2\theta$
$m=-3$				$\frac{1}{48}\sin^3\theta$	$\frac{1}{48}\sin^3\theta\cos\theta$
$m=-4$					$\frac{1}{384}\sin^4\theta$

Table 3: Table of associated Legendre functions $P_n^m(\cos\theta)$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$m=4$					$105\sin^4\theta\exp(4i\phi)/r^5$
$m=3$				$-15i\sin^3\theta\exp(3i\phi)/r^4$	$-105i\sin^3\theta\cos\theta\exp(3i\phi)/r^5$
$m=2$			$-3\sin^2\theta\exp(2i\phi)/r^3$	$-15\sin^2\theta\cos\theta\exp(2i\phi)/r^4$	$-\{105\cos^2\theta-15\}\sin^2\theta\times$ $\exp(2i\phi)/r^5$
$m=1$		$i\sin\theta\exp(i\phi)/r^2$	$3i\sin\theta\cos\theta\exp(i\phi)/r^3$	$3i\{5\cos^2\theta-1\}\sin\theta\exp(i\phi)/r^4$	$15i\cos\theta\{7\cos^2\theta-3\}\sin\theta\times$ $\exp(i\phi)/r^5$
$m=0$	$1/r$	$\cos(\theta)/r^2$	$\{3\cos^2\theta-1\}/r^3$	$6\{(\cos^3\theta)-\frac{3}{2}\sin^2\theta\cos\theta\}/r^4$	$24\{\cos^4\theta-3\sin^2\theta\cos^2\theta$ $+\frac{3}{8}\sin^4\theta\}/r^5$
$m=-1$		$i\sin\theta\exp(-i\phi)/r^2$	$3i\sin\theta\cos\theta\exp(-i\phi)/r^3$	$3i\{5\cos^2\theta-1\}\sin\theta\exp(-i\phi)/r^4$	$15i\cos\theta\{7\cos^2\theta-3\}\sin\theta\times$ $\exp(-i\phi)/r^5$
$m=-2$			$-3\sin^2\theta\exp(-2i\phi)/r^3$	$-15\sin^2\theta\cos\theta\exp(-2i\phi)/r^4$	$-\{105\cos^2\theta-15\}\sin^2\theta\times$ $\exp(-2i\phi)/r^5$
$m=-3$				$-15i\sin^3\theta\exp(-3i\phi)/r^4$	$-105i\sin^3\theta\cos\theta\exp(-3i\phi)/r^5$
$m=-4$					$105\sin^4\theta\exp(-4i\phi)/r^5$

Table 4: Table of outer harmonic functions $\mathcal{O}_n^m(\boldsymbol{x})$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$m=4$					$\frac{1}{384}\sin^4\theta\exp(4\mathrm{i}\,\phi)\,r^4$
$m=3$				$-\frac{1}{48}\mathrm{i}\,\sin^3\theta\exp(3\mathrm{i}\,\phi)\,r^3$	$\frac{1}{48}\mathrm{i}\,\sin^3\theta\cos\theta\exp(3\mathrm{i}\,\phi)\,r^4$
$m=2$			$-\frac{1}{8}\sin^2\theta\exp(2\mathrm{i}\,\phi)\,r^2$	$\frac{1}{8}\sin^2\theta\cos\theta\exp(2\mathrm{i}\,\phi)\,r^3$	$-\frac{1}{2}\left\{\frac{7}{48}\cos^2\theta-\frac{1}{48}\right\}\sin^2\theta\times$ $\exp(2\mathrm{i}\,\phi)\,r^4$
$m=1$		$\frac{1}{2}\mathrm{i}\,\sin\theta\exp(\mathrm{i}\,\phi)\,r$	$-\frac{1}{2}\mathrm{i}\,\sin\theta\cos\theta\exp(\mathrm{i}\,\phi)\,r^2$	$\frac{1}{16}\mathrm{i}\,\left\{5\cos^2\theta-1\right\}\sin\theta\exp(\mathrm{i}\,\phi)\,r^3$	$-\frac{1}{48}\mathrm{i}\,\cos\theta\left\{7\cos^2\theta-3\right\}\sin\theta\times$ $\exp(\mathrm{i}\,\phi)\,r^4$
$m=0$	1	$-\cos(\theta)\,r$	$\left\{\frac{3}{4}\cos^2\theta-\frac{1}{4}\right\}\,r^2$	$-\frac{1}{6}\left\{\cos^3\theta-\frac{3}{2}\sin^2\theta\cos\theta\right\}\,r^3$	$\frac{1}{24}\left\{\cos^4\theta-3\sin^2\theta\cos^2\theta\right.$ $\left.+\frac{3}{8}\sin^4\theta\right\}\,r^4$
$m=-1$		$\frac{1}{2}\mathrm{i}\,\sin\theta\exp(-\mathrm{i}\,\phi)\,r$	$-\frac{1}{2}\mathrm{i}\,\sin\theta\cos\theta\exp(-\mathrm{i}\,\phi)\,r^2$	$\frac{1}{16}\mathrm{i}\,\sin\theta\left\{5\cos^2\theta-1\right\}\exp(-\mathrm{i}\,\phi)\,r^3$	$-\frac{1}{48}\mathrm{i}\,\cos\theta\left\{7\cos^2\theta-3\right\}\sin\theta\times$ $\exp(-\mathrm{i}\,\phi)\,r^4$
$m=-2$			$-\frac{1}{8}\sin^2\theta\exp(-2\mathrm{i}\,\phi)\,r^2$	$\frac{1}{8}\sin^2\theta\cos\theta\exp(-2\mathrm{i}\,\phi)\,r^3$	$-\frac{1}{2}\left\{\frac{7}{48}\cos^2\theta-\frac{1}{48}\right\}\sin^2\theta\times$ $\exp(-2\mathrm{i}\,\phi)\,r^4$
$m=-3$				$-\frac{1}{48}\mathrm{i}\,\sin^3\theta\exp(-3\mathrm{i}\,\phi)\,r^3$	$\frac{1}{48}\mathrm{i}\,\sin^3\theta\cos\theta\exp(-3\mathrm{i}\,\phi)\,r^4$
$m=-4$					$\frac{1}{384}\sin^4\theta\exp(-4\mathrm{i}\,\phi)\,r^4$

Table 5: Table of inner harmonic functions $\mathcal{I}_n^m(\boldsymbol{x})$